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The Hydrodynamic Limit of a Deterministic Particle System with Conservation of Mass and Momentum

Michael G. Mürmann[1]

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We study the hydrodynamic limit of a deterministic one-dimensional particle system with nearest neighbour interaction and an additional regularizing force. Under its evolution mass and momentum are conserved. In the limit with Euler scaling their macroscopic distributions are shown to be governed by the compressible Navier–Stokes equations with a density dependent viscosity.

KEY WORDS: Interacting particle system; hydrodynamic limit; Euler scaling; conserved quantities; local equilibrium; compressible Navier–Stokes equations.

1. INTRODUCTION

With the aim to deduce the equations of hydrodynamics from microscopic dynamical systems, animated by the pioneering work of Guo, Papanicolaou and Varadhan,(4) but even before, several models have been studied rigorously. For a survey see Spohn(10) and Kipnis and Landim.(5) Most of these models are stochastic evolutions, since the random noise weakens the dependence on the initial conditions and provides good ergodic properties. Deterministic models were studied, e.g., by Mürmann(7, 8) and Uchiyama.(11, 12) They illustrate, that in principle it is possible to deduce hydrodynamic behaviour without the smoothing effect of noise. These models have only one conserved quantity, the particle number resp. mass, whose macroscopic dynamics under diffusive scaling is deduced, except for the system of hard rods of Boldrighini, Dobrushin and Sukhov(1) with Euler scaling and the opposite case of infinitely many conserved quantities,

[1] Institut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 294, D-69120 Heidelberg, Germany; e-mail: mmm@math.uni-heidelberg.de

in which velocities are transmitted and hence the number of particles with particular values of the velocity is conserved.

A stochastic lattice system with conservation of mass and momentum was studied by Esposito, Marra and Yau[3] and Quastel and Yau.[9] Its dynamics consists of an exclusion process superimposed by collisions that exchange velocities. They deduced the incompressible Navier–Stokes equations in the incompressible limit. These papers also contain a detailed discussion of the problem to deduce the Navier–Stokes equations from microscopic models.

In this paper we present a deterministic model, whose dynamics conserves mass and momentum. It is a one-dimensional system with nearest neighbour interaction and an additional velocity dependent force, which causes local equalization of velocities. It was stimulated by a similar model of Uchiyama.[11, 12] In the hydrodynamic limit with Euler scaling we deduce the compressible Navier–Stokes equations with a density dependent viscosity. We do not need to assume existence of solutions with given initial conditions, but obtain them as a consequence of the construction. For lack of uniqueness we cannot prove convergence, but the validity of the equation for any weak limit. Compactness results guarantee their existence.

Since the effect of the additional force tends to 0 with increasing distance of particles, we could not exclude a singular behaviour of the convection term at points of vanishing density. See ref. 6 for problems of the compressible Navier–Stokes equations at vacuum points. To avoid this we assume, that initially the density is strictly positive. For this reason we treat the system on the compact one-dimensional torus $\mathbf{T}$. We derive the validity of the compressible Navier–Stokes equations for the limit dynamics as long as the density remains strictly positive, which holds on a non-degenerate time interval. It is an open problem, whether the density of solutions with strictly positive initial density remains strictly positive (see ref. 6).

Furthermore we need regularity assumptions on the positions of the initial configurations, which also imply local equilibrium. We shall prove, that it is preserved in time and that for a.e. (almost every) macroscopic time local equilibrium holds for the velocities.

Though this model is quite different from the one we studied in refs. 7 and 8, some methods used there can be applied in modified form to this model, too. We have the energy as a Lyapunov functional and a further bounded functional resembling the energy decay in refs. 7 and 8. An essentially new problem is the study of the distribution of the momentum.

The paper is organized as follows. In the next section we introduce the model and formally deduce its limit dynamics. Then we derive inequalities for the functionals mentioned above. They are the basic tools for the proof of compactness in Section 4 and local equilibrium including regularity in

Section 5. In the last section these results serve to deduce the limit dynamics rigorously.

2. DESCRIPTION OF THE MODEL

We start from the classical one-dimensional Newtonian dynamics with nearest neighbour interaction with respect to a repulsive pair-potential Φ.

$$\frac{dx_i}{dt} = v_i \qquad (1 \leqslant i \leqslant N)$$

$$\frac{dv_i}{dt} = -F(x_{i+1} - x_i) + F(x_i - x_{i-1}) = \sum_{j:\,|j-i|=1} F(x_i - x_j)$$

with $F = -\Phi'$ and $x_i < x_{i+1}$ for $1 \leqslant i \leqslant N-1$.

The following assumptions on Φ are the same as in ref. 8.

$$\Phi\colon \mathbf{R} \setminus \{0\} \to \mathbf{R}^+$$

is twice continuously differentiable with the properties

1. symmetry: $\Phi(x) = \Phi(-x)$ for $x \neq 0$

2. convexity: there exists $0 < R \leqslant \infty$, such that F is strictly convex on $(0, R]$ and identically 0 on $[R, \infty)$, if $R < \infty$, resp. decreases to 0 as $x \to \infty$, if $R = \infty$.

3. singularity at 0:

(a) $\Phi(x) \to \infty$ as $|x| \to 0$

(b) there exists $\alpha, \gamma > 0$ such that $|x\Phi'(x)| \leqslant \alpha\Phi(x) + \gamma$ holds for $x \neq 0$.

Since Φ is convex, property 3b is equivalent, that $\Phi(x) \leqslant b\,|x|^{-\alpha}$ holds in a neighbourhood of 0 with a constant $b > 0$ (see, e.g., Dobrushin and Fritz[2]).

The singularity at 0 causes the order of the particles to be preserved in time, as can be shown by the boundedness of the energy (see Lemma 3.1).

We scale the microscopic space and time variables with Euler scaling, i.e., for $\varepsilon > 0$ we introduce the macroscopic positions

$$q_i(t) := \varepsilon x_i(\varepsilon^{-1} t) \qquad (1 \leqslant i \leqslant N)$$

with the corresponding velocities

$$p_i(t) := \frac{dq_i(t)}{dt} = v_i(\varepsilon^{-1} t)$$

We shall suppress the time variable from these and similar functions, unless we need the dependence on it explicitly.

The macroscopic variables evolve according to the system of equations

$$\frac{dq_i}{dt} = p_i$$

$$\frac{dp_i}{dt} = \varepsilon^{-1} \sum_{j:\,|j-i|=1} F\left(\frac{q_i - q_j}{\varepsilon}\right) \qquad (1 \leqslant i \leqslant N)$$

Even for this simplified form of classical dynamics an exact derivation of hydrodynamic behaviour at present seems to be inaccessible. In order to understand the transition from microscopic to macroscopic dynamics and to develop methods for its treatment, more or less sophisticated modifications have been introduced, which one is able to study rigorously. In our model we add a velocity dependent force, which resists the deviation of the velocities of neighboured particles and thus causes, as we shall see, local equalization of distances and velocities. This force is proportional to the difference of the velocities with a strength, which increases with decreasing distance of the particles with a singularity at 0 and vanishing at infinity. Because of the proportionality to the difference of the velocities the momentum is conserved. In the model of Uchiyama[11, 12] there is a similar resistance, which is proportional to the velocity itself and independent of the distances.

In the microscopic scale the effect of this force on the ith particle is given by

$$-\varepsilon^{-1} \sum_{j:\,|j-i|=1} (v_i - v_j)\, \chi(x_i - x_j)$$

where $\chi\colon \mathbf{R} \setminus \{0\} \to \mathbf{R}^+$ is continuously differentiable with the properties

1. symmetry: $\chi(x) = \chi(-x)$ for $x \neq 0$
2. convexity: χ is strictly convex on $(0, \infty)$
3. behaviour at 0 and ∞:

   (a) there exist $0 < c \leqslant C$ such that $\frac{c}{x^2} \leqslant \chi(x) \leqslant \frac{C}{x^2}$ holds for $x \neq 0$

   (b) there exists $\beta > 0$, such that $|x\chi'(x)| \leqslant \beta\, |\chi(x)|$ holds for $x \neq 0$

The lower bound of 3a is more substantial. It makes the regularizing effect sufficiently strong. The upper bound is needed to keep certain quantities finite.

From the remark to property 3 of Φ, applied to χ, one can easily conclude, that the upper bound of 3a implies 3b for bounded x with β depending on the bound. So 3b is essentially an additional requirement for large x. It holds, e.g., for $\chi(x) = \frac{c}{x^2}$.

An important consequence is the integrability at ∞ and non-integrability at 0.

As mentioned in the introduction we treat this model on the one-dimensional torus $\mathbf{T}$ with length 1, whose elements we represent by real numbers mod 1. In the sequel real numbers and intervals are always understood in this sense. Likewise the particles are arranged on $\mathbf{T}$ in the order $q_1, q_2, \dots, q_N, q_{N+1}$ with $q_{N+1} = q_1$.

Eventually we arrived at the final equations of the macroscopic variables

$$\frac{dq_i}{dt} = p_i \tag{2.1a}$$

$$\frac{dp_i}{dt} = \varepsilon^{-1} \sum_{j:\,|j-i|=1} F\left(\frac{q_i - q_j}{\varepsilon}\right) - \varepsilon^{-2} \sum_{j:\,|j-i|=1} (p_i - p_j)\, \chi\left(\frac{q_i - q_j}{\varepsilon}\right) \tag{2.1b}$$

Since the additional force even in the microscopic scale contains a scaling parameter, the limit we are going to study is not really a scaling limit. This is indeed impossible for a system with a Navier–Stokes limit dynamics, which is not scaling invariant. This force seems to be of higher order of magnitude as the classical force. But one can interpret its scaling as a scaling of the difference of the velocities with ε^{-1}, and in local equilibrium this force will turn out to be of finite order, too. Out of local equilibrium however it should drive the system into local equilibrium. Unfortunately we can only prove this for the velocities.

Since we assume, that on the microscopic scale the number of particles in bounded intervals is of finite order, on the macroscopic scale it diverges of order ε^{-1}. Therefore we scale the mass of the particles with ε and represent the configurations by their following macroscopic distributions of mass and momentum

$$\rho_t^\varepsilon := \varepsilon \sum_i \delta_{q_i(t)}$$

$$v_t^\varepsilon := \varepsilon \sum_i p_i(t)\, \delta_{q_i(t)}$$

As we consider the system with finite macroscopic mass, N is of order ε^{-1}, too. This normalization coincides with the empirical distributions up to a factor of finite order.

We study the evolution of these distributions by applying them to test functions $\varphi \in C^1$, the space of continuously differentiable functions on $\mathbf{T}$. (2.1) becomes

$$\frac{d}{dt}\int \varphi\, d\rho_t^\varepsilon = \varepsilon \sum_i \varphi'(q_i)\, p_i = \int \varphi'\, dv_t^\varepsilon \tag{2.2a}$$

$$\begin{aligned}\frac{d}{dt}\int \varphi\, dv_t^\varepsilon &= \varepsilon \sum_i \varphi'(q_i)\, p_i^2 + \varepsilon \sum_i \varphi(q_i)\, \varepsilon^{-1} \sum_{j:\,|j-i|=1} F\left(\frac{q_i-q_j}{\varepsilon}\right) \\ &\quad - \varepsilon \sum_i \varphi(q_i)\, \varepsilon^{-2} \sum_{j:\,|j-i|=1} (p_i-p_j)\, \chi\left(\frac{q_i-q_j}{\varepsilon}\right) \\ &= \varepsilon \sum_i \varphi'(q_i)\, p_i^2 + \frac{\varepsilon}{2} \sum_{i,\,j:\,|j-i|=1} \frac{\varphi(q_i)-\varphi(q_j)}{\varepsilon}\, F\left(\frac{q_i-q_j}{\varepsilon}\right) \\ &\quad - \frac{\varepsilon}{2} \sum_{i,\,j:\,|j-i|=1} \frac{\varphi(q_i)-\varphi(q_j)}{\varepsilon} \cdot \frac{p_i-p_j}{\varepsilon}\, \chi\left(\frac{q_i-q_j}{\varepsilon}\right)\end{aligned} \tag{2.2b}$$

The transformations are valid due to the symmetry properties of F and χ. This representation indicates, that the effect of the classical force is of finite order of magnitude. This also holds for the additional force in case of local equilibrium, since then the difference of the velocities of neighboured particles will be shown to be of order ε.

Setting particularly $\varphi \equiv 1$ implies conservation of mass and momentum.

At the end of this section we want to deduce the limit dynamics in a formal way. For it we assume, that ρ_t^ε and v_t^ε have weak limits ρ_t resp. v_t such that ρ_t has a sufficiently smooth density, which we likewise denote by ρ_t, and v_t has a sufficiently smooth density with respect to ρ_t, which we denote by u_t. Then in the limit (2.2a) obviously becomes the continuity equation

$$\frac{\partial}{\partial t}\rho_t(q) = -\frac{\partial}{\partial q}(\rho_t(q)\, u_t(q))$$

in a weak sense (see Eq. (6.1a) for the exact formulation).

For the determination of the limit of (2.2b) we study the terms on the right-hand side separately. We assume local equilibrium, which means, that the distances and velocities of the particles are locally constant. Then at time t near q the microscopic distances $x_{i+1}-x_i = \frac{q_{i+1}-q_i}{\varepsilon}$ are approximately $\frac{1}{\rho_t(q)}$ and the velocities $u_t(q)$. Concerning the second term this leads to $\int \varphi'(q)\, F(\frac{1}{\rho_t(q)})\, dq$. The factor $\frac{1}{\rho_t(q)}$ cancels, since we have to integrate with respect to ρ_t. As we assume the velocities to be locally constant, the analogy to the variance of a random variable suggests, that in the limit the density of the distribution of the square of the velocities with respect to ρ_t is the square of the density of the velocity distribution. So we obtain $\int \varphi'(q)\, \rho_t(q)\, u_t(q)^2\, dq$ for the first term. To determine the limit of the third term we sum $\frac{p_{i+1}-p_i}{\varepsilon}$ with $q_i \in [q, q+\delta)$. u_t being sufficiently smooth, $\frac{p_{i+1}-p_i}{\varepsilon}$ is locally approximately $\frac{1}{\rho_t(q)}\, u_t'(q)$ and the third term becomes $-\int \varphi'(q)\, \frac{1}{\rho_t(q)}^2\, \chi(\frac{1}{\rho_t(q)})\, u_t'(q)\, \rho_t(q)\, dq$.

We thus heuristically deduced the equation

$$\frac{\partial}{\partial t}\,(\rho_t(q)\, u_t(q))$$
$$= -\frac{\partial}{\partial q}\,(\rho_t(q)\, u_t(q)^2) - \frac{\partial}{\partial q}\, F\left(\frac{1}{\rho_t(q)}\right) + \frac{\partial}{\partial q}\left(\frac{1}{\rho_t(q)}\, \chi\left(\frac{1}{\rho_t(q)}\right) \frac{\partial}{\partial q}\, u_t(q)\right)$$

in a weak sense (see Eq. (6.1b)).

With the continuity equation these are the compressible Navier–Stokes equations with pressure $F(\frac{1}{\rho})$ and density dependent viscosity $\frac{1}{\rho}\,\chi(\frac{1}{\rho})$. In virtue of property 3a of χ the latter is of linear order in ρ. Since we treated them on the torus, they can also be conceived as periodic solutions on $\mathbf{R}$.

For a clearer representation of the equations of the limit dynamics we deviate from our general notation and set $\rho(t, q) = \rho_t(q)$ and $u(t, q) = u_t(q)$. Then the equations take the familiar form

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial q}\,(\rho u)$$
$$\frac{\partial}{\partial t}\,(\rho u) = -\frac{\partial}{\partial q}\,(\rho u^2) - \frac{\partial}{\partial q}\, p(\rho) + \frac{\partial}{\partial q}\left(\mu(\rho)\, \frac{\partial u}{\partial q}\right)$$

with $p(\rho) = F(\frac{1}{\rho})$ and $\mu(\rho) = \frac{1}{\rho}\,\chi(\frac{1}{\rho})$.

3. BASIC INEQUALITIES

In this section we introduce two important functionals of the configuration. We show, that their boundedness is preserved in time, which is fundamental for the following. The first is the energy defined by

$$E^{\varepsilon} := \varepsilon \sum_i \left(\frac{p_i^2}{2} + \frac{1}{2} \sum_{j:\,|j-i|=1} \Phi\left(\frac{q_i - q_j}{\varepsilon}\right)\right)$$

By the equations of motion (2.1) its time derivative is

$$\frac{dE^{\varepsilon}(t)}{dt} = \varepsilon \sum_i p_i \left[\varepsilon^{-1} \sum_{j:\,|j-i|=1} F\left(\frac{q_i - q_j}{\varepsilon}\right) - \varepsilon^{-2} \sum_{j:\,|j-i|=1} (p_i - p_j)\,\chi\left(\frac{q_i - q_j}{\varepsilon}\right)\right]$$

$$-\frac{\varepsilon}{2} \sum_{i,\,j:\,|j-i|=1} F\left(\frac{q_i - q_j}{\varepsilon}\right) \cdot \frac{p_i - p_j}{\varepsilon}$$

$$= -\frac{\varepsilon}{2} \sum_{i,\,j:\,|j-i|=1} \chi\left(\frac{q_i - q_j}{\varepsilon}\right) \cdot \left(\frac{p_i - p_j}{\varepsilon}\right)^2 \leqslant 0$$

which yields

Lemma 3.1. $E^{\varepsilon}(t) \geqslant 0$ is decreasing with

$$\frac{dE^{\varepsilon}(t)}{dt} = -\frac{\varepsilon}{2} \sum_{i,\,j:\,|j-i|=1} \chi\left(\frac{q_i - q_j}{\varepsilon}\right) \cdot \left(\frac{p_i - p_j}{\varepsilon}\right)^2$$

In the sequel we occasionally understand

$$\frac{dE^{\varepsilon}}{dt} = -\frac{\varepsilon}{2} \sum_{i,\,j:\,|j-i|=1} \chi\left(\frac{q_i - q_j}{\varepsilon}\right) \cdot \left(\frac{p_i - p_j}{\varepsilon}\right)^2$$

as a functional of the configuration, even if we consider it without reference to time.

For the definition of the second functional we first introduce

$$X\colon \mathbf{R} \setminus \{0\} \to \mathbf{R}$$

defined by

$$X(x) := -\int_x^{\infty} \chi(y)\,dy \qquad \text{for} \quad x > 0$$

with $X(x) = -X(-x)$ for $x < 0$ such that $X'(x) = \chi(x)$ holds for $x \neq 0$.

$-X$ has similar properties as F and will take its place in some estimates of refs. 7 and 8. The second basic functional of the configuration is defined by

$$\Xi^{\varepsilon} := \varepsilon \sum_i \left(\sum_{j:\,|j-i|-1} \varepsilon^{-1} X\left(\frac{q_i - q_j}{\varepsilon}\right)\right)^2$$

For its estimation we calculate the derivative

$$\frac{d}{dt}\left[\sum_{i,\,j:\,|j-i|=1} X\left(\frac{q_i-q_j}{\varepsilon}\right)p_i\right]$$

$$=\varepsilon^{-1}\sum_{i,\,j:\,|j-i|=1} \chi\left(\frac{q_i-q_j}{\varepsilon}\right)(p_i-p_j)\,p_i$$

$$+\sum_{i,\,j:\,|j-i|=1} X\left(\frac{q_i-q_j}{\varepsilon}\right)\sum_{k:\,|k-i|=1}\left[\varepsilon^{-1}F\left(\frac{q_i-q_k}{\varepsilon}\right)-\varepsilon^{-2}(p_i-p_k)\,\chi\left(\frac{q_i-q_k}{\varepsilon}\right)\right]$$

$$=\frac{\varepsilon}{2}\sum_{i,\,j:\,|j-i|=1} \chi\left(\frac{q_i-q_j}{\varepsilon}\right)\left(\frac{p_i-p_j}{\varepsilon}\right)^2$$

$$+\varepsilon^{-1}\sum_i\left[\left(\sum_{j:\,|j-i|=1} X\left(\frac{q_i-q_j}{\varepsilon}\right)\right)\left(\sum_{k:\,|k-i|=1} F\left(\frac{q_i-q_k}{\varepsilon}\right)\right)\right]$$

$$-\frac{d}{dt}\frac{\varepsilon}{2}\sum_i\left(\sum_{j:\,|j-i|=1}\varepsilon^{-1}X\left(\frac{q_i-q_j}{\varepsilon}\right)\right)^2$$

$$\leqslant -\left(\frac{dE^\varepsilon}{dt}+\frac{1}{2}\frac{d\Xi^\varepsilon}{dt}\right)$$

The inequality holds, since X and F are monotone on $(0, \infty)$ in the opposite sense.

It follows

$$\frac{d\Xi^\varepsilon}{dt}+2\frac{dE^\varepsilon}{dt}+2\frac{d}{dt}\left[\sum_{i,\,j:\,|j-i|=1} X\left(\frac{q_i-q_j}{\varepsilon}\right)p_i\right]\leqslant 0$$

hence

$$\Xi^\varepsilon(t)+2E^\varepsilon(t)+2\left[\sum_{i,\,j:\,|j-i|=1} X\left(\frac{q_i-q_j}{\varepsilon}\right)p_i\right]$$

is decreasing.

With the inequality

$$\left|\sum_{i,\,j:\,|j-i|=1} X\left(\frac{q_i-q_j}{\varepsilon}\right)p_i\right|\leqslant(\Xi^\varepsilon)^{1/2}\left(\varepsilon\sum_i p_i^2\right)^{1/2}\leqslant(2E^\varepsilon\Xi^\varepsilon)^{1/2}$$

we get for $t > s$

$$\Xi^\varepsilon(t) \leqslant \Xi^\varepsilon(s) + 2E^\varepsilon(s) - 2E^\varepsilon(t) + 2\sqrt{2}[(E^\varepsilon(s)\,\Xi^\varepsilon(s))^{1/2} + (E^\varepsilon(t)\,\Xi^\varepsilon(t))^{1/2}]$$

Since $x \leqslant a\sqrt{x} + b$ with x, a , $b \geqslant 0$ implies $x \leqslant (\frac{a}{2} + \sqrt{\frac{a^2}{4} + b})^2 \leqslant a^2 + 2b$, we obtain with simple estimations and the monotonicity of the energy the following lemma.

Lemma 3.2. $\Xi^\varepsilon(t) \leqslant (2 + 2\sqrt{2})\,\Xi^\varepsilon(s) + (8 + 2\sqrt{2})\,E^\varepsilon(s)$ holds for $t > s \geqslant 0$.

4. COMPACTNESS

In this section we prove tightness of the distributions of mass and momentum and convergence of subsequences for all times with weakly continuous limits.

We assume, that for $0 < \varepsilon \leqslant \varepsilon_0$ we are given initial configurations $(\rho_0^\varepsilon, v_0^\varepsilon)$, which evolve according to the Eqs. (2.2a, b). Since the space of the position is compact, the tightness of the distribution of mass is equivalent to the boundedness of the total mass, which is preserved in time. Besides the tightness of the initial distributions of mass we need the boundedness of the energy.

Theorem 4.1. If $\{E^\varepsilon(0): 0 < \varepsilon \leqslant \varepsilon_0\}$ is bounded and $\{\rho_0^\varepsilon: 0 < \varepsilon \leqslant \varepsilon_0\}$ is tight, then also $\{\rho_t^\varepsilon: 0 < \varepsilon \leqslant \varepsilon_0, t \geqslant 0\}$ and $\{v_t^\varepsilon: 0 < \varepsilon \leqslant \varepsilon_0, t \geqslant 0\}$ are tight. For every sequence $\varepsilon_n \to 0$ there exists a subsequence $\varepsilon_{n(k)}$, such that $\rho_t^{\varepsilon_{n(k)}}$ and $v_t^{\varepsilon_{n(k)}}$ weakly converge for every $t \geqslant 0$ as $\varepsilon_{n(k)} \to 0$. The limit measures $\{\rho_t, t \geqslant 0\}$ are absolutely continuous with respect to the Lebesgue measure and the limit signed measures $\{v_t, t \geqslant 0\}$ are absolutely continuous with respect to the corresponding measures $\{\rho_t, t \geqslant 0\}$. Both are weakly continuous in t.

Proof. As remarked above the tightness of $\{\rho_t^\varepsilon: 0 < \varepsilon \leqslant \varepsilon_0, t \geqslant 0\}$ is obvious. With the boundedness of the energy this would also hold in **R** (see refs. 7 or 8) for compact time intervals.

Let E be an upper bound of the initial energy and hence of the energy at every time and M be an upper bound of the total mass.

The tightness yields for a given sequence the weak convergence of a subsequence at the times of a denumerable dense subset of $[0, \infty)$. The convergence for all times and the weak continuity follows from

$$\left|\frac{d}{dt}\int \varphi\, d\rho_t^\varepsilon\right| = \left|\varepsilon \sum_i \varphi'(q_i(t))\, p_i(t)\right|$$
$$\leqslant \left(\varepsilon \sum_i \varphi'(q_i(t))^2\right)^{1/2} \left(\varepsilon \sum_i p_i(t)^2\right)^{1/2} \leqslant (2ME)^{1/2}\, \|\varphi'\|$$

for $\varphi \in C^1$ with $\|\psi\| := \sup\{|\psi(q)| : q \in \mathbf{T}\}$ denoting the uniform norm.

The proof of the absolute continuity of the limit measures $\{\rho_t, t \geqslant 0\}$ in ref. 7 (Theorem 3.1) only uses the boundedness of the potential energy and thus is valid here, too.

The tightness of $\{v_t^\varepsilon : 0 < \varepsilon \leqslant \varepsilon_0, t \geqslant 0\}$ follows from the tightness of $\{\rho_t^\varepsilon : 0 < \varepsilon \leqslant \varepsilon_0, t \geqslant 0\}$ and

$$|v_t^\varepsilon|\,(C) = \varepsilon \sum_i 1_C(q_i(t))\, |p_i(t)| \leqslant \left(\varepsilon \sum_i 1_C(q_i(t))\right)^{1/2} \left(\varepsilon \sum_i p_i(t)^2\right)^{1/2}$$
$$\leqslant (2E\rho_t^\varepsilon(C))^{1/2}$$

This inequality also shows, that limit distributions of momentum are absolutely continuous with respect to the corresponding distribution of mass.

For the convergence for all times and the weak continuity we estimate the terms of the right-hand side of (2.2b) separately.

$$\left|\varepsilon \sum_i \varphi'(q_i)\, p_i^2\right| \leqslant 2E\, \|\varphi'\|$$

$$\left|\frac{\varepsilon}{2} \sum_{i,j:\,|j-i|=1} \frac{\varphi(q_i)-\varphi(q_j)}{\varepsilon}\, F\left(\frac{q_i-q_j}{\varepsilon}\right)\right|$$
$$\leqslant \|\varphi'\|\, \frac{\varepsilon}{2} \sum_{i,j:\,|j-i|=1} \left|\frac{q_i-q_j}{\varepsilon}\, F\left(\frac{q_i-q_j}{\varepsilon}\right)\right|$$
$$\leqslant \|\varphi'\|\, (\alpha E + \gamma M)$$

$$\left|\varepsilon \sum_{i,j:\,|j-i|=1} \frac{\varphi(q_i)-\varphi(q_j)}{\varepsilon}\, \varepsilon^{-1} p_i \chi\left(\frac{q_i-q_j}{\varepsilon}\right)\right|$$
$$= \left|\frac{\varepsilon}{2} \sum_{i,j:\,|j-i|=1} \frac{\varphi(q_i)-\varphi(q_j)}{\varepsilon} \cdot \frac{p_i-p_j}{\varepsilon}\, \chi\left(\frac{q_i-q_j}{\varepsilon}\right)\right|$$
$$\leqslant \|\varphi'\|\, \frac{\varepsilon}{2} \sum_{i,j:\,|j-i|=1} \left|\frac{q_i-q_j}{\varepsilon}\right| \cdot \left|\frac{p_i-p_j}{\varepsilon}\right| \chi\left(\frac{q_i-q_j}{\varepsilon}\right)$$

We do not estimate this term uniformly, but its integral, what is indeed only needed.

$$\int_r^t \frac{\varepsilon}{2} \sum_{i,j:\,|j-i|=1} \left|\frac{q_i(s)-q_j(s)}{\varepsilon}\right| \cdot \left|\frac{p_i(s)-p_j(s)}{\varepsilon}\right| \chi\left(\frac{q_i(s)-q_j(s)}{\varepsilon}\right) ds$$

$$\leqslant \left(\int_r^t \frac{\varepsilon}{2} \sum_{i,j:\,|j-i|=1} \left(\frac{p_i(s)-p_j(s)}{\varepsilon}\right)^2 \chi\left(\frac{q_i(s)-q_j(s)}{\varepsilon}\right) ds\right)^{1/2}$$

$$\times \left(\int_r^t \frac{\varepsilon}{2} \sum_{i,j:\,|j-i|=1} \left(\frac{q_i(s)-q_j(s)}{\varepsilon}\right)^2 \chi\left(\frac{q_i(s)-q_j(s)}{\varepsilon}\right) ds\right)^{1/2}$$

$$\leqslant [EMC(t-r)]^{1/2}$$

5. LOCAL EQUILIBRIUM

For the derivation of local equilibrium we consider time independent configurations with $\rho^\varepsilon \to \rho$ and $v^\varepsilon \to v$ weakly as $\varepsilon \to 0$. We shall identify the measure ρ with its density and denote the density of v with respect to ρ by u. In addition to the boundedness of the energy we need the boundedness of $\{\Xi^\varepsilon: 0 < \varepsilon \leqslant \varepsilon_0\}$.

The derivation of local equilibrium of the configuration of the positions can be proved the same way as in refs. 7 and 8. For a self-contained presentation we briefly recapitulate the procedure and the needed results adapted to the present model.

One first has to notice, that in refs. 7 and 8 the velocity v_i is not an independent variable, but a function of the configuration of the positions. Recall further, that $-X$ takes the place of F. This means that we replace v_i in refs. 7 and 8 by

$$-\varepsilon^{-1} \sum_{j:\,|j-i|=1} X\left(\frac{q_i-q_j}{\varepsilon}\right)$$

and hence v^ε by

$$\mu^\varepsilon := \varepsilon \sum_i \left(-\varepsilon^{-1} \sum_{j:\,|j-i|=1} X\left(\frac{q_i-q_j}{\varepsilon}\right)\right) \delta_{q_i}$$

An important inequality is the following (Lemma 2.3 of ref. 8).

For every interval $I = [a, b)$ and n, m with $q_n, q_m \in I$ there holds

$$\left| X\left(\frac{q_{n+1}-q_n}{\varepsilon}\right) - X\left(\frac{q_{m+1}-q_m}{\varepsilon}\right) \right|^2 \leqslant \rho^\varepsilon(I)\, \varepsilon \sum_{i:\, q_i \in I} \left(\sum_{j:\, |j-i|=1} \varepsilon^{-1} X\left(\frac{q_i-q_j}{\varepsilon}\right)\right)^2 \tag{5.1}$$

This inequality first yields, that ρ is bounded.

With a suitable choice of q_n and q_m and a limit procedure one obtains

$$\frac{1}{\rho(I)} \left| X\left(\frac{1}{\rho(b)}\right) - X\left(\frac{1}{\rho(a)}\right) \right|^2 \leqslant \varliminf_{\varepsilon \to 0} \varepsilon \sum_{i:\, q_i \in I} \left(\sum_{j:\, |j-i|=1} \varepsilon^{-1} X\left(\frac{q_i-q_j}{\varepsilon}\right)\right)^2$$

Obviously the left-hand side and similar expressions below have to be interpreted as 0 in the case $\rho(I) = 0$. The same holds analogously for functions like $\frac{1}{\rho}\, \chi(\frac{1}{\rho})$ at $\rho = 0$.

The application of this inequality to disjoint intervals yields, that ρ can be chosen such that $X(\frac{1}{\rho})$ is an absolutely continuous function.

By replacing $\frac{q_{m+1}-q_m}{\varepsilon}$ in (5.1) with their mean value one gets for $q_n \in I$

$$\left| X\left(\frac{q_{n+1}-q_n}{\varepsilon}\right) - X\left(\frac{|I^\varepsilon|}{\rho^\varepsilon(I)}\right) \right|^2 \leqslant \rho^\varepsilon(I)\, \varepsilon \sum_{i:\, q_i \in I} \left(\sum_{j:\, |j-i|=1} \varepsilon^{-1} X\left(\frac{q_i-q_j}{\varepsilon}\right)\right)^2 \tag{5.2}$$

where I^ε arises from I by slightly changing its boundary points to the position of the respective next particle to the right. Without restriction one can assume, that its length converges to that of I (see ref. 8) as $\varepsilon \to 0$.

To determine the limit of μ^ε we set

$$d^\varepsilon(q) = \frac{q_{n+1}-q_n}{\varepsilon} \quad \text{with } n = n(q) \text{ such that } q_n < q \leqslant q_{n+1} \text{ holds, hence}$$

$$\mu^\varepsilon([a, b)) = X(d^\varepsilon(b)) - X(d^\varepsilon(a)) \text{ for } a < b$$

With $n = n(a)$ resp. $n(b)$ and suited intervals in (5.2) one proves weak convergence of μ^ε to the signed measure with density $\frac{\partial}{\partial q} X(\frac{1}{\rho(q)})$ with respect to the Lebesgue measure. In contrast to ref. 8 there is a different force in the second term of (2.2b) than in the definition of μ^ε. But this additional difficulty can easily be resolved. We set

$$\overline{\mu^\varepsilon} := \varepsilon \sum_i \left(\varepsilon^{-1} \sum_{j:\, |j-i|=1} F\left(\frac{q_i-q_j}{\varepsilon}\right)\right) \delta_{q_i}$$

such that $\overline{\mu^\varepsilon}([a,b)) = -F(d^\varepsilon(b)) + F(d^\varepsilon(a))$ holds. The continuous differentiability of the inverse function of $(-X)$ yields weak convergence of $\overline{\mu^\varepsilon}$ to the signed measure with density $-\frac{\partial}{\partial q} F(\frac{1}{\rho(q)})$ with respect to the Lebesgue measure. In the same way ρ itself is shown to be an absolutely continuous function.

We summarize the results.

Theorem 5.1. Let $\rho^\varepsilon \to \rho$ weakly as $\varepsilon \to 0$ with bounded $\{E^\varepsilon: 0 < \varepsilon \leqslant \varepsilon_0\}$ and $\{\Xi^\varepsilon: 0 < \varepsilon \leqslant \varepsilon_0\}$. Then the density ρ can be chosen such that ρ is bounded and ρ and $F(\frac{1}{\rho})$ are absolutely continuous functions. $\overline{\mu^\varepsilon}$ weakly converges to the signed measure with density $-\frac{\partial}{\partial q} F(\frac{1}{\rho(q)})$ with respect to the Lebesgue measure.

If these assumptions hold, we shall tacitly assume this version of ρ in the sequel.

Next we prove, that the velocities are locally constant. For it we need the boundedness of the time derivative of the energy.

The following lemma provides the basic inequality.

Lemma 5.2. For bounded $\{\Xi^\varepsilon: 0 < \varepsilon \leqslant \varepsilon_0\}$ there exist $\bar{C} > 0$ and $\theta > 0$, such that for every interval $I = [a, b)$ and $\varepsilon > 0$ with $\rho^\varepsilon(I) \geqslant \theta\, |I^\varepsilon|^2$

$$\left(\varepsilon \sum_{i:\, q_i \in I} \left|\frac{p_{i+1} - p_i}{\varepsilon}\right|\right)^2$$

$$\leqslant \bar{C} \rho^\varepsilon(I) \left[\chi\left(\frac{|I^\varepsilon|}{\rho^3(I)}\right)\right]^{-1} \varepsilon \sum_{i:\, q_i \in I} \left(\frac{p_{i+1} - p_i}{\varepsilon}\right)^2 \chi\left(\frac{q_{i+1} - q_i}{\varepsilon}\right)$$

$\bar{C}$ only depends on χ, whereas θ also depends on a bound of $\{\Xi^\varepsilon: 0 < \varepsilon \leqslant \varepsilon_0\}$.

Corollary 5.3. Under the assumptions of Lemma 5.2 there holds for $q_n, q_m \in I$

$$|p_n - p_m|^2 \leqslant \bar{C} \rho^\varepsilon(I) \left[\chi\left(\frac{|I^\varepsilon|}{\rho^\varepsilon(I)}\right)\right]^{-1} \varepsilon \sum_{i:\, q_i \in I} \left(\frac{p_{i+1} - p_i}{\varepsilon}\right)^2 \chi\left(\frac{q_{i+1} - q_i}{\varepsilon}\right)$$

Proof. For simplicity we first treat the special case $\chi(x) = \frac{1}{x^2}$ with $X(x) = -\frac{1}{x}$.

We start with the estimate

$$\begin{aligned}
&\varepsilon \sum_{i:\, q_i \in I} \left|\frac{p_{i+1}-p_i}{\varepsilon}\right| \left(\frac{q_{i+1}-q_i}{\varepsilon}\right)^{-1} \\
&\quad \leqslant \left(\varepsilon \sum_{i:\, q_i \in I} 1\right)^{1/2} \left(\varepsilon \sum_{i:\, q_i \in I} \left(\frac{p_{i+1}-p_i}{\varepsilon}\right)^2 \left(\frac{q_{i+1}-q_i}{\varepsilon}\right)^{-2}\right)^{1/2} \\
&\quad = (\rho^\varepsilon(I))^{1/2} \left(\varepsilon \sum_{i:\, q_i \in I} \left(\frac{p_{i+1}-p_i}{\varepsilon}\right)^2 \chi\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)\right)^{1/2}
\end{aligned}$$

From (5.2) we get

$$\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)^{-1} \geqslant \frac{\rho^\varepsilon(I)}{|I^\varepsilon|} - (\rho^\varepsilon(I)\, \Xi)^{1/2} \qquad \text{for} \quad q_i \in I$$

where Ξ is an upper bound of $\{\Xi^\varepsilon: 0 < \varepsilon \leqslant \varepsilon_0\}$.

If $\rho^\varepsilon(I) \geqslant 4\Xi\, |I^\varepsilon|^2$ holds, it follows $(\frac{q_{i+1}-q_i}{\varepsilon})^{-1} \geqslant \frac{1}{2}\frac{\rho^\varepsilon(I)}{|I^\varepsilon|}$ and hence

$$\varepsilon \sum_{i:\, q_i \in I} \left|\frac{p_{i+1}-p_i}{\varepsilon}\right| \left(\frac{q_{i+1}-q_i}{\varepsilon}\right)^{-1} \geqslant \frac{1}{2}\frac{\rho^\varepsilon(I)}{|I^\varepsilon|}\, \varepsilon \sum_{i:\, q_i \in I} \left|\frac{p_{i+1}-p_i}{\varepsilon}\right|$$

Combining this inequality with the first one, the lemma follows with $\theta = 4\Xi$ and $\bar{C} = 4$. For general χ the inequalities can be traced back to the special case by means of property 3a of χ, which by integration yields corresponding estimates for X and its differences. The inequalities and consequently θ and $\bar{C}$ then contain the constants c and C from property 3a of χ.

The corollary easily follows from

$$|p_n - p_m|^2 \leqslant \left(\varepsilon \sum_{i:\, q_i \in I} \left|\frac{p_{i+1}-p_i}{\varepsilon}\right|\right)^2$$

We shall apply Lemma 5.2 and Corollary 5.3 in the case, that ess inf $\rho > 0$, which for continuous ρ is equivalent to $\rho > 0$. Then the condition of Lemma 5.2 is satisfied for sufficiently small ε and $|I|$. More precisely, let $\rho \geqslant \eta > 0$ and I be an interval with $|I| \leqslant \frac{\eta}{3\theta}$. Then $\frac{\rho^\varepsilon(I)}{|I^\varepsilon|} \geqslant \frac{\eta}{2}$ and $|I^\varepsilon| \leqslant \frac{\eta}{2\theta}$ hold for ε sufficiently small, which implies $\rho^\varepsilon(I) \geqslant \theta\, |I^\varepsilon|^2$. Evidently this also holds, if $\rho \geqslant \eta$ only on I. In the sequel the assumption sufficiently small ε and $|I|$ always refers to this.

We now derive local equilibrium of the velocities in the sense as stated in the following lemma.

Lemma 5.4. Let $\rho^\varepsilon \to \rho$ weakly as $\varepsilon \to 0$ with bounded $\{\Xi^\varepsilon : 0 < \varepsilon \leqslant \varepsilon_0\}$ and $\{\frac{dE^\varepsilon}{dt} : 0 < \varepsilon \leqslant \varepsilon_0\}$ and ess inf $\rho > 0$. Then there converges for partitions of $\mathbf{T}$ into disjoint subintervals $I_k = [a_k, b_k)$ $(1 \leqslant k \leqslant K)$ with mean velocity $\overline{p_k} = \frac{v^\varepsilon(I_k)}{\rho^\varepsilon(I_k)}$ in the interval I_k

$$\lim_{\varepsilon \to 0} \sum_k \varepsilon \sum_{i:\, q_i \in I_k} |p_i - \overline{p_k}|^2 \to 0 \qquad \text{as} \quad \sup_k |I_k| \to 0.$$

Proof. Let I and ε satisfy $\rho^\varepsilon(I) \geqslant \theta\, |I^\varepsilon|^2$. Replacing p_m in Corollary 5.3 with its mean

$$\bar{p} = \frac{\varepsilon \sum_{i:\, q_i \in I} p_i}{\varepsilon \sum_{i:\, q_i \in I} 1} = \frac{v^\varepsilon(I)}{\rho^\varepsilon(I)}$$

one gets for $q_n \in I$

$$|p_n - \bar{p}|^2 \leqslant \bar{C} \rho^\varepsilon(I) \left[\chi\left(\frac{|I^\varepsilon|}{\rho^\varepsilon(I)} \right) \right]^{-1} \varepsilon \sum_{i:\, q_i \in I} \left(\frac{p_{i+1} - p_i}{\varepsilon} \right)^2 \chi\left(\frac{q_{i+1} - q_i}{\varepsilon} \right)$$

and hence

$$\varepsilon \sum_{i:\, q_i \in I} |p_i - \bar{p}|^2 \leqslant \bar{C} \rho^\varepsilon(I)^2 \left[\chi\left(\frac{|I^\varepsilon|}{\rho^\varepsilon(I)} \right) \right]^{-1} \varepsilon \sum_{i:\, q_i \in I} \left(\frac{p_{i+1} - p_i}{\varepsilon} \right)^2 \chi\left(\frac{q_{i+1} - q_i}{\varepsilon} \right)$$

We decompose $\mathbf{T}$ into sufficiently small disjoint subintervals $I_k = [a_k, b_k)$ $(1 \leqslant k \leqslant K)$ and apply the inequality to the subintervals for sufficiently small ε. With $\overline{p_k}$ denoting the mean in the interval I_k it follows

$$\begin{aligned} &\sum_k \varepsilon \sum_{i:\, q_i \in I_k} |p_i - \overline{p_k}|^2 \\ &\quad \leqslant \bar{C} \sum_k \rho^\varepsilon(I_k)^2 \left[\chi\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)} \right) \right]^{-1} \varepsilon \sum_{i:\, q_i \in I_k} \left(\frac{p_{i+1} - p_i}{\varepsilon} \right)^2 \chi\left(\frac{q_{i+1} - q_i}{\varepsilon} \right) \\ &\quad \leqslant \bar{C} \left(-\frac{dE^\varepsilon}{dt} \right) \sup_k \left(\rho^\varepsilon(I_k)^2 \left[\chi\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)} \right) \right]^{-1} \right) \end{aligned}$$

We fix the partition and let $\varepsilon \to 0$. Then

$$\sup_k \left(\rho^\varepsilon(I_k)^2 \left[\chi\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)} \right) \right]^{-1} \right) \to \sup_k \left(\rho(I_k)^2 \left[\chi\left(\frac{|I_k|}{\rho(I_k)} \right) \right]^{-1} \right)$$

Now let $\sup_k |I_k| \to 0$. Then due to property 3a of χ

$$\sup_k \left(\rho(I_k)^2 \left[\chi\left(\frac{|I_k|}{\rho(I_k)} \right) \right]^{-1} \right) \to 0$$

finishing the proof of Lemma 5.4.

As a consequence of it we show, that for convergent v^ε the distribution of the square of the velocities weakly converges to the measure with density, which is the square of the corresponding density of the velocity distribution, as described in the heuristic deduction of the limit dynamics. For it we need the boundedness of the energy.

We denote this distribution by

$$\kappa^\varepsilon := \varepsilon \sum_i p_i^2 \delta_{q_i}$$

We decompose an arbitrary interval $I = [a, b)$ into sufficiently small disjoint subintervals $I_k = [a_k, b_k)$ $(1 \leqslant k \leqslant K)$.

Since by definition of $\overline{p_k}$

$$\sum_{i:\, q_i \in I_k} (p_i - \overline{p_k}) = 0$$

we can represent

$$\kappa^\varepsilon(I_k) = \varepsilon \sum_{i:\, q_i \in I_k} \overline{p_k}^2 + \varepsilon \sum_{i:\, q_i \in I_k} (p_i - \overline{p_k})^2 = \rho^\varepsilon(I_k)\, \overline{p_k}^2 + \varepsilon \sum_{i:\, q_i \in I_k} (p_i - \overline{p_k})^2$$

$$\kappa^\varepsilon(I) = \sum_k \rho^\varepsilon(I_k) \left(\frac{v^\varepsilon(I_k)}{\rho^\varepsilon(I_k)} \right)^2 + \sum_k \varepsilon \sum_{i:\, q_i \in I_k} (p_i - \overline{p_k})^2$$

As $\varepsilon \to 0$ the first term of the right-hand side converges to $\sum_k \rho(I_k)(\frac{v(I_k)}{\rho(I_k)})^2$. From $\kappa^\varepsilon(I) \leqslant 2E^\varepsilon$ we conclude $\sum_k \rho(I_k)(\frac{v(I_k)}{\rho(I_k)})^2 \leqslant 2E$, if E is a bound of the energy.

Finally let $\sup_k |I_k| \to 0$. We determine the limit of $\sum_k \rho(I_k)(\frac{v(I_k)}{\rho(I_k)})^2$ by considering the martingale with respect to the normalized restriction of ρ to I, indexed by these partitions, which attributes to each partition the function $\sum_k \frac{v(I_k)}{\rho(I_k)} 1_{I_k}$. By the inequality above it is bounded in L^2 by $2E$ and hence converges in L^2. It follows

$$\sum_k \rho(I_k) \left(\frac{v(I_k)}{\rho(I_k)} \right)^2 \to \int_I u(q)^2\, \rho(q)\, dq \leqslant 2E.$$

The second term converges to 0 in the same sense by Lemma 5.4.

Since the partitions are only auxiliary tools, on which $\kappa^\varepsilon(I)$ does not depend, there easily follows the convergence $\kappa^\varepsilon(I) \to \int_I u(q)^2\, \rho(q)\, dq$ as $\varepsilon \to 0$.

With $I = \mathbf{T}$ also $\int u(q)^2\, \rho(q)\, dq \leqslant 2E < \infty$ holds.

We thus proved the following corollary.

Corollary 5.5. Let the assumptions of Lemma 5.4 be satisfied with bounded $\{E^\varepsilon : 0 < \varepsilon \leqslant \varepsilon_0\}$ and let ν^ε weakly converge to a signed measure with density u with respect to ρ as $\varepsilon \to 0$. Then the measures κ^ε weakly converge to the measure with density u^2 with respect to ρ. Its total mass is bounded by a bound of $\{2E^\varepsilon : 0 < \varepsilon \leqslant \varepsilon_0\}$.

Only for Lemma 5.4 and Corollary 5.5 we need the boundedness of ρ from below. Without this assumption one could modify the proof and derive the result on $\{\rho > 0\}$ by proceeding as in the proof of the next theorem. But this does not exclude the possibility, that in the limit κ^ε has singular parts on $\{\rho = 0\}$.

It remains to derive smoothness of the density u. For it we additionally need the smoothness properties of ρ from Theorem 5.1, hence we require its assumptions.

Theorem 5.6. Let $\rho^\varepsilon \to \rho$ and $\nu^\varepsilon \to \nu$ weakly as $\varepsilon \to 0$ with bounded $\{E^\varepsilon : 0 < \varepsilon \leqslant \varepsilon_0\}$ and $\{\Xi^\varepsilon : 0 < \varepsilon \leqslant \varepsilon_0\}$ and finite $\underline{\lim}_{\varepsilon \to 0}(-\frac{dE^\varepsilon}{dt})$. Then the density u of ν with respect to ρ is ρ-a.e. differentiable with $u' \in L^2(\rho)$ and

$$\int u'(q)^2 \left(\frac{1}{\rho(q)}\right)^2 \chi\left(\frac{1}{\rho(q)}\right) \rho(q)\, dq \leqslant \bar{C}\, \underline{\lim}_{\varepsilon \to 0} \left(-\frac{dE^\varepsilon}{dt}\right)$$

Remark 1. We did not simplify the left-hand side by reducing a factor ρ in order to emphasize, that it is in fact an integral with respect to ρ (see property 3a of χ and the proof). In particular the integral is well-defined.

Remark 2. If we reconsider the time evolution of configurations in the next section, the boundedness of the energy implies by Fatou's Lemma, that $\underline{\lim}_{\varepsilon \to 0}(-\frac{dE^\varepsilon(t)}{dt})$ is finite for a.e. t. In this way this assumption will be satisfied.

Proof. Let $I = [a, b)$ and ε satisfy $\rho^\varepsilon(I) \geqslant \theta\, |I^\varepsilon|^2$ and let $0 < \delta < \frac{b-a}{2}$. Assume without restriction

$$\frac{\nu^\varepsilon([a, a+\delta))}{\rho^\varepsilon([a, a+\delta))} \geqslant \frac{\nu^\varepsilon([b-\delta, b))}{\rho^3([b-\delta, b))}.$$

Since $\nu^\varepsilon([a, a+\delta)) = \varepsilon \sum_{i:\, q_i \in [a, a+\delta)} p_i$, there exists $q_n \in [a, a+\delta)$ with

$$p_n \geqslant \frac{\nu^\varepsilon([a, a+\delta))}{\rho^\varepsilon([a, a+\delta))}.$$

Equally there exists $q_m \in [b-\delta, b)$ with

$$p_m \leqslant \frac{\nu^\varepsilon([b-\delta, b))}{\rho^\varepsilon([b-\delta, b))}.$$

Inserting this p_n and p_m into Corollary 5.3 yields

$$\left|\frac{\nu^\varepsilon([a, a+\delta))}{\rho^\varepsilon([a, a+\delta))} - \frac{\nu^\varepsilon([b-\delta, b))}{\rho^\varepsilon([b-\delta, b))}\right|^2 \frac{1}{\rho^\varepsilon(I)}\, \chi\left(\frac{|I^\varepsilon|}{\rho^\varepsilon(I)}\right)$$
$$\leqslant \bar{C}\varepsilon \sum_{i:\, q_i \in I} \left(\frac{p_{i+1}-p_i}{\varepsilon}\right)^2 \chi\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)$$

We apply this inequality to sufficiently small disjoint intervals $I_k = [a_k, b_k)$ $(1 \leqslant k \leqslant K)$ on which $\rho \geqslant \eta > 0$ and sufficiently small ε. It follows

$$\sum_k \left|\frac{\nu^\varepsilon([a_k, a_k+\delta))}{\rho^\varepsilon([a_k, a_k+\delta))} - \frac{\nu^\varepsilon([b_k-\delta, b_k))}{\rho^\varepsilon([b_k-\delta, b_k))}\right|^2 \frac{1}{\rho^\varepsilon(I_k)}\, \chi\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\right) \leqslant -\bar{C}\, \frac{dE^\varepsilon}{dt}$$

With a sequence $\varepsilon_n \to 0$ such that

$$\left(-\frac{dE^{\varepsilon_n}}{dt}\right) \to \varliminf_{\varepsilon \to 0} \left(-\frac{dE^\varepsilon}{dt}\right)$$

we get

$$\sum_k \left|\frac{\nu([a_k, a_k+\delta))}{\rho([a_k, a_k+\delta))} - \frac{\nu([b_k-\delta, b_k))}{\rho([b_k-\delta, b_k))}\right|^2 \frac{1}{\rho(I_k)}\, \chi\left(\frac{|I_k|}{\rho(I_k)}\right) \leqslant \bar{C} \varliminf_{\varepsilon \to 0} \left(-\frac{dE^\varepsilon}{dt}\right)$$

Since ν is absolutely continuous with respect to ρ with density u,

$$\frac{\nu([q, q+\delta))}{\rho([q, q+\delta))} \to u(q)$$

for ρ-a.e. q as $\delta \to 0$ and it follows for such intervals $I_k = [a_k, b_k)$ for ρ-a.e. a_k, b_k

$$\sum_k |u(b_k) - u(a_k)|^2 \frac{1}{\rho(I_k)} \chi\left(\frac{|I_k|}{\rho(I_k)}\right) \leqslant \bar{C} \varliminf_{\varepsilon \to 0} \left(-\frac{dE^\varepsilon}{dt}\right) \tag{5.3}$$

Recall that we choose the absolutely continuous version of ρ. For fixed $q \in \mathbf{T}$ with $\rho(q) > 0$ there exists an interval $I = [a, b)$ with $a < q < b$ and $\eta > 0$ such that $\rho \geqslant \eta$ on I. Then (5.3) holds for sufficiently small disjoint subintervals of I yielding

$$\begin{aligned} &\sum_k |u(b_k) - u(a_k)| \\ &\leqslant \left(\sum_k |u(b_k) - u(a_k)|^2 \frac{1}{\rho(I_k)} \chi\left(\frac{|I_k|}{\rho(I_k)}\right)\right)^{1/2} \left(\sum_k \rho(I_k) \left[\chi\left(\frac{|I_k|}{\rho(I_k)}\right)\right]^{-1}\right)^{1/2} \\ &\leqslant \left(\bar{C} \varliminf_{\varepsilon \to 0} \left(-\frac{dE^\varepsilon}{dt}\right)\right)^{1/2} \chi\left(\frac{1}{\eta}\right)^{-1/2} \left(\sum_k \rho(I_k)\right)^{1/2} \end{aligned}$$

Since ρ is absolutely continuous with respect to the Lebesgue measure, it follows, that u restricted to I is an absolutely continuous function and hence differentiable a.e. with respect to the Lebesgue measure. By the choice of I the last property extends to $\{\rho > 0\}$, so u is ρ-a.e. differentiable.

For the proof of the inequality we fix $\eta > 0$. There exists a finite cover of the compact set $\{\rho \geqslant 2\eta\}$ with open intervals of length $\leqslant \frac{\eta}{3\theta}$ and boundary points outside the null set of (5.3), on which $\rho > \eta$. By adding their left boundary points we obtain a finite cover of $\{\rho \geqslant 2\eta\}$ with disjoint intervals $I_k = [a_k, b_k)$ $(1 \leqslant k \leqslant K)$, on which $\rho \geqslant \eta$. These intervals satisfy the conditions for (5.3). We represent its left-hand side as

$$\sum_k \left|\frac{u(b_k) - u(a_k)}{b_k - a_k}\right|^2 \left(\frac{|I_k|}{\rho(I_k)}\right)^2 \chi\left(\frac{|I_k|}{\rho(I_k)}\right) \rho(I_k),$$

which is the integral of the function

$$\sum_k \left|\frac{u(b_k) - u(a_k)}{b_k - a_k}\right|^2 \left(\frac{|I_k|}{\rho(I_k)}\right)^2 \chi\left(\frac{|I_k|}{\rho(I_k)}\right) 1_{I_k}$$

with respect to ρ over the union of the $I_k = [a_k, b_k)$.

Let $\sup_k |I_k| \to 0$ along a sequence of partitions.
Then

$$\sum_k \left|\frac{u(b_k)-u(a_k)}{b_k-a_k}\right|^2 \left(\frac{|I_k|}{\rho(I_k)}\right)^2 \chi\left(\frac{|I_k|}{\rho(I_k)}\right) 1_{I_k} \to u'^2 \left(\frac{1}{\rho}\right)^2 \chi\left(\frac{1}{\rho}\right) \rho\text{-a.e.}$$

on $\{\rho \geqslant 2\eta\}$ and it follows with (5.3) and Fatou's lemma

$$\int_{\{\rho \geqslant 2\eta\}} u'(q)^2 \left(\frac{1}{\rho(q)}\right)^2 \chi\left(\frac{1}{\rho(q)}\right) \rho(q)\, dq \leqslant \bar{C} \varliminf_{\varepsilon \to 0} \left(-\frac{dE^\varepsilon}{dt}\right)$$

Since we integrate with respect to ρ, the inequality of Theorem 5.6 follows with $\eta \downarrow 0$. The lower bound of property 3a of χ finally yields $u' \in L^2(\rho)$.

6. THE LIMIT DYNAMICS

In the preceding section we deduced the limit behaviour of configurations without reference to time. Let us now return to the dynamics (2.2a, b) and apply these results to the configurations at fixed times. We prove, that under suitable conditions of the initial configurations any weak limit solves the compressible Navier–Stokes equations formally deduced in Section 2 in a time interval, on which the density is strictly positive. For convergent initial configurations convergence to a solution holds for subsequences.

We first make the following assumptions

$$\rho_t^\varepsilon \to \rho_t \quad \text{and} \quad v_t^\varepsilon \to v_t \quad \text{weakly for all} \quad t \geqslant 0 \quad \text{as} \quad \varepsilon \to 0$$

$$\{E^\varepsilon(0) : 0 < \varepsilon \leqslant \varepsilon_0\} \quad \text{and} \quad \{\Xi^\varepsilon(0) : 0 < \varepsilon \leqslant \varepsilon_0\} \quad \text{are bounded.}$$

By Lemmas 3.1 and 3.2 these quantities are uniformly bounded for times $t \geqslant 0$. Concerning the boundedness of the energy, its assumption is usual and not too restrictive. But the results in Section 5 show, that the boundedness of $\{\Xi^\varepsilon(0)\colon 0 < \varepsilon \leqslant \varepsilon_0\}$ can only be satisfied, if the positions and the limit density are already initially quite regular. The initial configurations of the velocities however are only restricted by the boundedness of the kinetic energy. Their local equalization then follows for a.e. time from the boundedness of the time derivative of the energy.

According to our notation we identify ρ_t with its density and denote the density of v_t with respect to ρ_t by u_t.

Let $0 < T \leqslant \infty$ to be determined later. For $0 < t < T$ we integrate the Eqs. (2.2a, b) over the interval $[0, t]$ and let $\varepsilon \to 0$.

Since for Eq. (2.2a) the integrand is uniformly bounded in ε (see Section 4), we are allowed to permute the limit with the integral and obtain the continuity equation

$$\int \varphi(q)\,\rho_t(q)\,dq = \int \varphi(q)\,\rho_0(q)\,dq + \int_0^t \left[\int \varphi'(q)\,u_s(q)\,\rho_s(q)\,dq\right] ds \tag{6.1a}$$

For the treatment of (2.2b) we assume, that $\rho_0 > 0$. Then there exists $\tau > 0$, such that $\rho_t > 0$ for $0 \leqslant t < \tau$. To show this, let $\rho_0 \geqslant \eta > 0$ and Ξ be an upper bound of $\{\Xi^\varepsilon(t)\colon 0 < \varepsilon \leqslant \varepsilon_0,\, t \geqslant 0\}$. Then

$$-X\left(\frac{|I|}{\rho_0(I)}\right) \geqslant -X\left(\frac{1}{\eta}\right)$$

holds for every interval I and we can divide $\mathbf{T}$ into disjoint intervals $I_k = [a_k, b_k)$ $(1 \leqslant k \leqslant K)$ with

$$-X\left(\frac{|I_k|}{\rho_0(I_k)}\right) - (\Xi\rho_0(I_k))^{1/2} \geqslant -\frac{1}{2}X\left(\frac{1}{\eta}\right)$$

for $1 \leqslant k \leqslant K$. By weak continuity of ρ_t there exists $\tau > 0$ with

$$-X\left(\frac{|I_k|}{\rho_s(I_k)}\right) - (\Xi\rho_s(I_k))^{1/2} \geqslant -\frac{1}{4}X\left(\frac{1}{\eta}\right)$$

for $0 \leqslant s \leqslant \tau$. Finally (5.2) with $\varepsilon \to 0$ implies

$$-X\left(\frac{1}{\rho_s(q)}\right) \geqslant -\frac{1}{4}X\left(\frac{1}{\eta}\right)$$

for $q \in \mathbf{T}$, hence $\inf_{0 \leqslant s \leqslant \tau} \rho_s > 0$. By the same argument applied to t with $\rho_t > 0$ there exists a neighbourhood of t, in which ρ_s is bounded from below by a strictly positive constant. Compactness yields it for $0 \leqslant s \leqslant t < T$, if $\rho_t > 0$ for $0 \leqslant t < T$. In the sequel we suppose T to have this property.

We treat the terms of the right-hand side of (2.2b) separately.

Let $\varphi \in C^1$. For a fixed time $0 \leqslant s \leqslant t$ the first term converges by Corollary 5.5 to

$$\int \varphi'(q)\,u_s(q)^2\,\rho_s(q)\,dq$$

Again by uniform boundedness we permute the limit with the integral and get

$$\int_0^t \left[\int \varphi'(q)\, u_s(q)^2\, \rho_s(q)\, dq \right] ds$$

The second term converges by Theorem 5.1 to

$$-\int \varphi(q) \frac{\partial}{\partial q} F\left(\frac{1}{\rho_s(q)}\right) dq = \int \varphi'(q)\, F\left(\frac{1}{\rho_s(q)}\right) dq$$

The same reasoning as for the first term leads to

$$\int_0^t \left[\int \varphi'(q)\, F\left(\frac{1}{\rho_s(q)}\right) dq \right] ds$$

The third term is more difficult to handle. We represent it as

$$\begin{aligned} -\varepsilon \sum_i & \frac{\varphi(q_{i+1}) - \varphi(q_i)}{\varepsilon} \cdot \frac{p_{i+1} - p_i}{\varepsilon}\, \chi\left(\frac{q_{i+1} - q_i}{\varepsilon}\right) \\ &= -\varepsilon \sum_i \varphi'(\widetilde{q}_i)\, \frac{q_{i+1} - q_i}{\varepsilon} \cdot \frac{p_{i+1} - p_i}{\varepsilon}\, \chi\left(\frac{q_{i+1} - q_i}{\varepsilon}\right) \end{aligned}$$

with $q_i < \widetilde{q}_i < q_{i+1}$.

Let $I_k = [a_k, b_k)$ $(1 \leqslant k \leqslant K)$ be a partition of $\mathbf{T}$ into disjoint subintervals with an arbitrary choice $\overline{q_k} \in I_k$. Then

$$\begin{aligned} -\varepsilon \sum_i & \varphi'(\widetilde{q}_i)\, \frac{q_{i+1} - q_i}{\varepsilon} \cdot \frac{p_{i+1} - p_i}{\varepsilon}\, \chi\left(\frac{q_{i+1} - q_i}{\varepsilon}\right) \qquad (6.2) \\ &= -\varepsilon \sum_k \sum_{i:\, q_i \in I_k} (\varphi'(\widetilde{q}_i) - \varphi'(\overline{q_k}))\, \frac{q_{i+1} - q_i}{\varepsilon}\, \chi\left(\frac{q_{i+1} - q_i}{\varepsilon}\right) \frac{p_{i+1} - p_i}{\varepsilon} \\ &\quad - \varepsilon \sum_k \varphi'(\overline{q_k}) \sum_{i:\, q_i \in I_k} \left[\frac{q_{i+1} - q_i}{\varepsilon}\, \chi\left(\frac{q_{i+1} - q_i}{\varepsilon}\right) - \frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\, \chi\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\right) \right] \frac{p_{i+1} - p_i}{\varepsilon} \\ &\quad - \sum_k \varphi'(\overline{q_k})\, \frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\, \chi\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\right) \left[\varepsilon \sum_{i:\, q_i \in I_k} \frac{p_{i+1} - p_i}{\varepsilon} \right] \end{aligned}$$

We first show, that the integral of the first two terms of the right-hand side over $[0, t]$ tend to 0 as $\varepsilon \to 0$ followed by $\sup_k |I_k| \to 0$.

With the modulus of continuity $\omega(\delta)=\sup\{|\varphi'(q_1)-\varphi'(q_2)|\colon |q_1-q_2|\leqslant\delta\}$ of φ' the absolute value of the first term is bounded by

$$\omega(\sup_k |I_k|)\,\varepsilon\sum_k \sum_{i:\,q_i\in I_k}\left|\frac{q_{i+1}-q_i}{\varepsilon}\right|\cdot\left|\frac{p_{i+1}-p_i}{\varepsilon}\right|\chi\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)$$

Since the integral of

$$\varepsilon\sum_k \sum_{i:\,q_i\in I_k}\left|\frac{q_{i+1}-q_i}{\varepsilon}\right|\cdot\left|\frac{p_{i+1}-p_i}{\varepsilon}\right|\chi\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)$$

over $[0,t]$ is bounded by the last inequality in Section 4, the asserted convergence of the first term of (6.2) follows.

For the treatment of the second term we consider the function $g(x):=x\chi(x)$ for $x\neq 0$. From $g'(x)=x\chi'(x)+\chi(x)$ and property 3b of χ it follows $|\frac{g'}{X'}|=|\frac{g'}{\chi}|\leqslant\beta+1$ and hence

$$\begin{aligned}&\left|\frac{q_{i+1}-q_i}{\varepsilon}\chi\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)-\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\chi\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\right)\right|\\&\qquad\leqslant(\beta+1)\left|X\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)-X\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\right)\right|\end{aligned}$$

In the sequel we assume, that the subintervals and ε are sufficiently small, that we can apply Lemma 5.2. By the uniform boundedness of the density from below this is simultaneously possible for $0\leqslant s\leqslant t$. Then we can estimate the absolute value of the second term of (6.2) up to a constant factor by

$$\begin{aligned}&\varepsilon\sum_k \sum_{i:\,q_i\in I_k}\left|X\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)-X\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\right)\right|\cdot\left|\frac{p_{i+1}-p_i}{\varepsilon}\right|\\
&\leqslant\varepsilon\sum_k\left[\sum_{i:\,q_i\in I_k}\left|\frac{p_{i+1}-p_i}{\varepsilon}\right|\right]\cdot\left[\rho^\varepsilon(I_k)\,\varepsilon\sum_{l:\,q_l\in I_k}\left(\sum_{j:\,|j-l|=1}\varepsilon^{-1}X\left(\frac{q_l-q_j}{\varepsilon}\right)\right)^2\right]^{1/2}\\
&\leqslant\left[\sum_k\varepsilon\sum_{i:\,q_i\in I_k}\left(\sum_{j:\,|j-i|=1}\varepsilon^{-1}X\left(\frac{q_i-q_j}{\varepsilon}\right)\right)^2\right]^{1/2}\\
&\quad\times\left[\sum_k\rho^\varepsilon(I_k)\left(\varepsilon\sum_{i:\,q_i\in I_k}\left|\frac{p_{i+1}-p_i}{\varepsilon}\right|\right)^2\right]^{1/2}\\
&\leqslant c^{-1/2}\bar{C}(\Xi^\varepsilon)^{1/2}\left(-\frac{dE^\varepsilon}{dt}\right)^{1/2}(\sup_k|I_k|)\end{aligned}$$

The first inequality holds by (5.2) and the last by Lemma 5.2 with the lower bound of property 3a of χ. If we integrate over $[0, t]$ and apply Cauchy's inequality to the integral, we obtain with Lemma 3.2 the upper bound

$$c^{-1/2}\bar{C}[((2+2\sqrt{2})\,\Xi^{\varepsilon}(0)+(8+2\sqrt{2})\,E^{\varepsilon}(0))\,E^{\varepsilon}(0)\,t]^{1/2}\,(\sup_k |I_k|)$$

which implies the convergence of the second term of (6.2).

It remains to determine the limit of the decisive third term.

From the convergence

$$\frac{\nu([q, q+\delta))}{\rho([q, q+\delta))} \to u(q) \qquad \text{for} \quad \rho\text{-a.e.} \quad q$$

as $\delta \to 0$ and Corollary 5.3 applied to $[q, q+\delta)$ with p_m replaced by $\bar{p}$ as in the proof of Lemma 5.4 one can easily conclude, that as $\varepsilon \to 0$ for ρ-a.e. a_k, b_k $(1 \leqslant k \leqslant K)$

$$\varepsilon \sum_{i:\, q_i \in I_k} \frac{p_{i+1}-p_i}{\varepsilon} \to u(b_k)-u(a_k)$$

$$-\sum_k \varphi'(\overline{q_k})\,\frac{|I_k^{\varepsilon}|}{\rho^{\varepsilon}(I_k)}\,\chi\left(\frac{|I_k^{\varepsilon}|}{\rho^{\varepsilon}(I_k)}\right)\left(\varepsilon \sum_{i:\, q_i \in I_k} \frac{p_{i+1}-p_i}{\varepsilon}\right)$$

$$\to -\sum_k \varphi'(\overline{q_k})\,\frac{|I_k|}{\rho(I_k)}\,\chi\left(\frac{|I_k|}{\rho(I_k)}\right)(u(b_k)-u(a_k))$$

Recall, that in general we suppress the time variable. One has to notice here however, that the choice of the a_k, b_k depends on time. Since these functions appear in dependence on time only as integrands, we choose a_k, b_k such that the times, for which the convergence does not hold, has Lebesgue measure 0. This choice is justified by regarding the measure obtained by integrating ρ_s with respect to the Lebesgue measure.

To permute the limit with the integral over $[0, t]$ we show, that

$$-\sum_k \varphi'(\overline{q_k})\,\frac{|I_k^{\varepsilon}|}{\rho^{\varepsilon}(I_k)}\,\chi\left(\frac{|I_k^{\varepsilon}|}{\rho^{\varepsilon}(I_k)}\right)\left(\varepsilon \sum_{i:\, q_i \in I_k} \frac{p_{i+1}-p_i}{\varepsilon}\right)$$

is uniformly L^2-bounded and therefore uniformly integrable. This follows from the estimate, again with Lemma 5.2

$$\begin{aligned}
&\left(\sum_k \frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\chi\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\right)\left|\varepsilon\sum_{i:\,q_i\in I_k}\frac{p_{i+1}-p_i}{\varepsilon}\right|\right)^2\\
&\leqslant\left(\sum_k\left[\bar{C}\frac{|I_k^\varepsilon|^2}{\rho^\varepsilon(I_k)}\chi\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\right)\varepsilon\sum_{i:\,q_i\in I_k}\left(\frac{p_{i+1}-p_i}{\varepsilon}\right)^2\chi\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)\right]^{1/2}\right)^2\\
&\leqslant\bar{C}\left[\sum_k\frac{|I_k^\varepsilon|^2}{\rho^\varepsilon(I_k)}\chi\left(\frac{|I_k^\varepsilon|}{\rho^\varepsilon(I_k)}\right)\right]\left(-\frac{dE^\varepsilon}{dt}\right)\\
&\leqslant C\bar{C}\left(\sum_k\rho^\varepsilon(I_k)\right)\left(-\frac{dE^\varepsilon}{dt}\right)\leqslant C\bar{C}M\left(-\frac{dE^\varepsilon}{dt}\right)
\end{aligned}$$

Its integral is bounded by $C\bar{C}ME$, if E is an upper bound of the energy. Now let $\sup_k |I_k|\to 0$ along sequences.

For fixed s we represent

$$-\sum_k \varphi'(\overline{q_k})\frac{|I_k|}{\rho_s(I_k)}\chi\left(\frac{|I_k|}{\rho_s(I_k)}\right)(u_s(b_k)-u_s(a_k))$$

as the integral of the function

$$-\sum_k \varphi'(\overline{q_k})\left(\frac{|I_k|}{\rho_s(I_k)}\right)^2\chi\left(\frac{|I_k|}{\rho_s(I_k)}\right)\left(\frac{u_s(b_k)-u_s(a_k)}{b_k-a_k}\right)1_{I_k}$$

with respect to ρ_s.

From Theorem 5.6 (see Remark 2) follows the convergence

$$\begin{aligned}
&-\sum_k \varphi'(\overline{q_k})\left(\frac{|I_k|}{\rho_s(I_k)}\right)^2\chi\left(\frac{|I_k|}{\rho_s(I_k)}\right)\left(\frac{u_s(b_k)-u_s(a_k)}{b_k-a_k}\right)1_{I_k}\\
&\qquad\to -\varphi'\cdot\left(\frac{1}{\rho_s}\right)^2\chi\left(\frac{1}{\rho_s}\right)u_s'
\end{aligned}$$

a.e. with respect to ρ_s integrated with respect to the Lebesgue measure on $[0, t]$. We show, that the corresponding integrals converge.

For this purpose we prove again uniform L^2-boundedness. The integral of the square of

$$-\sum_k \varphi'(\overline{q_k})\left(\frac{|I_k|}{\rho_s(I_k)}\right)^2\chi\left(\frac{|I_k|}{\rho_s(I_k)}\right)\left(\frac{u_s(b_k)-u_s(a_k)}{b_k-a_k}\right)1_{I_k}$$

with respect to ρ_s is up to a constant factor bounded by the left-hand side of (5.3) (use property 3a of χ), whose integral over $[0, t]$ is uniformly bounded due to the right-hand side of (5.3).

We finally proved, that the integral of the third term of (2.2b) over $[0, t]$ converges as $\varepsilon \to 0$ to

$$-\int_0^t \left[\int \varphi'(q) \left(\frac{1}{\rho_s(q)} \right)^2 \chi \left(\frac{1}{\rho_s(q)} \right) u_s'(q) \, \rho_s(q) \, dq \right] ds$$

With the obvious limit of the integral of the left-hand side of (2.2b) we obtain the second equation of the limit dynamics, to which we recall (6.1a) for a joint presentation.

$$\int \varphi(q) \, \rho_t(q) \, dq = \int \varphi(q) \, \rho_0(q) \, dq + \int_0^t \left[\int \varphi'(q) \, u_s(q) \, \rho_s(q) \, dq \right] ds \tag{6.1a}$$

$$\int \varphi(q) \, u_t(q) \, \rho_t(q) \, dq - \int \varphi(q) \, u_0(q) \, \rho_0(q) \, dq$$
$$= \int_0^t \left[\int \varphi'(q) \left(\rho_s(q) \, u_s(q)^2 + F\left(\frac{1}{\rho_s(q)} \right) - \frac{1}{\rho_s(q)} \chi \left(\frac{1}{\rho_s(q)} \right) u_s'(q) \right) dq \right] ds \tag{6.1b}$$

Here we effected the simplification corresponding to Remark 1 to Theorem 5.1.

We finally proved

Theorem 6.1. Let $\rho_t^\varepsilon \to \rho_t$ and $v_t^\varepsilon \to v_t$ weakly for $t \geqslant 0$ as $\varepsilon \to 0$ with bounded $\{E^\varepsilon(0): 0 < \varepsilon \leqslant \varepsilon_0\}$ and $\{\Xi^\varepsilon(0): 0 < \varepsilon \leqslant \varepsilon_0\}$ and $\rho_0 > 0$. Then there exists $0 < T \leqslant \infty$ with $\rho_t > 0$ for $0 \leqslant t < T$. Denoting the density of v_t with respect to ρ_t by u_t the limit distributions satisfy the equations

$$\frac{\partial}{\partial t} \rho_t(q) = -\frac{\partial}{\partial q} (\rho_t(q) \, u_t(q))$$

$$\frac{\partial}{\partial t} (\rho_t(q) \, u_t(q)) = -\frac{\partial}{\partial q} (\rho_t(q) \, u_t(q)^2) - \frac{\partial}{\partial q} F\left(\frac{1}{\rho_t(q)} \right)$$
$$+ \frac{\partial}{\partial q} \left(\frac{1}{\rho_t(q)} \chi \left(\frac{1}{\rho_t(q)} \right) \frac{\partial}{\partial q} u_t(q) \right)$$

in the weak sense (6.1a, b) on the interval $[0, T)$.

For a concise representation of the equations recall the form at the end of Section 2.

Theorem 6.1 signifies, that under the quoted conditions any limit dynamics solves the Eqs. (6.1a, b). For the proof of a limit theorem, which only assumes convergence and suitable boundedness conditions of the initial configurations, one needs the unique solvability of (6.1a, b) with given initial conditions. Then it would easily follow from Theorem 6.1 by means of Theorem 4.1. For lack of uniqueness these results at least yield convergence of subsequences to the limit dynamics, as stated in the last theorem.

Theorem 6.2. Let $\rho_0^{\varepsilon_n} \to \rho_0$ and $v_0^{\varepsilon_n} \to v_0$ weakly as $\varepsilon_n \to 0$ with bounded $\{E^{\varepsilon_n}(0): n \geqslant 1\}$ and $\{\Xi^{\varepsilon_n}(0): n \geqslant 1\}$ and $\rho_0 > 0$. Then there exists a subsequence $\varepsilon_{n(k)}$, such that $\rho_t^{\varepsilon_{n(k)}}$ and $v_t^{\varepsilon_{n(k)}}$ weakly converge for every $t \geqslant 0$. The limit distributions (ρ_t, v_t) satisfy the equations stated in Theorem 6.1 with initial conditions (ρ_0, v_0) on the interval $[0, T)$ with $0 < T \leqslant \infty$, on which $\rho_t > 0$.

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